J. Math. Anal. Appl. 354 (2009) 412–420

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Subordination results for classes of analytic functions related to conic domains defined by a fractional operator

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ARTICLE INFO

Article history:
Received 8 October 2007
Available online 21 October 2008
Submitted by I. Podlubny

Keywords:
Uniformly convex function
Fractional derivative
Subordination
Conic domain
Koebe domain

ABSTRACT

Let a fractional operator $D_\lambda^{n,\alpha}$ ($n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $0 \leqslant \alpha < 1$, $\lambda \geqslant 0$) be defined by

$$\begin{aligned}
&D_\lambda^{0,0} = f(z),\\
&D_\lambda^{1,\alpha} f(z) = (1-\lambda)\Omega^\alpha f(z) + \lambda z\big(\Omega^\alpha f(z)\big)' = D_\lambda^\alpha\big(f(z)\big),\\
&D_\lambda^{2,\alpha} f(z) = D_\lambda^\alpha\big(D_\lambda^{1,\alpha} f(z)\big),\\
&\quad\vdots\\
&D_\lambda^{n,\alpha} f(z) = D_\lambda^\alpha\big(D_\lambda^{n-1,\alpha} f(z)\big),
\end{aligned}$$

where

$$\Omega^\alpha f(z) = \Gamma(2-\alpha)z^\alpha D_z^\alpha f(z),$$

and D_z^α is the known fractional derivative. In this paper, several interesting subordination results are derived for certain classes of analytic functions related to conic domains defined by the operator $D_\lambda^{n,\alpha}$, which yield sharp distortion, rotation theorems and Koebe domain. These results extended corresponding previously known results.

1. Introduction

Let $E = \{z \in \mathbb{C}\colon\ |z| < 1\}$ be the open unit disk in complex plane $\mathbb{C}$ and let A be a class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

analytic in the unit disk E.

For functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.10.025

the Hadamard product (or convolution) $f * g$ is defined, as usual, by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

An analytic function f in E is said to be subordinate to an analytic function g in E (written $f \prec g$), if $f(z) = g(w(z))$, $z \in E$, for some analytic function w with $w(0) = 0$ and $|w(z)| < 1$, $z \in E$. In particular, if the function g is univalent, then the above subordination is equivalent to $f(0) = g(0)$ and $f(E) \subseteq g(E)$.

Let $CV(\gamma)$ and $ST(\gamma)$ denote the usual classes of convex and starlike functions of order γ, $0 \leqslant \gamma < 1$, respectively.

Let $UCV(\beta, \gamma)$ be the class of uniformly convex functions f of order γ and type β. We say that $f \in A$ is in $UCV(\beta, \gamma)$ [4] if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right| + \gamma,$$

where $\beta \geqslant 0$, $0 \leqslant \gamma < 1$, and is said to be in the class $SP(\beta, \gamma)$ [4] if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma.$$

It is obvious that $f \in UCV(\beta, \gamma)$ if and only if $zf' \in SP(\beta, \gamma)$.

Each of the following definitions will also be required in our present investigation.

The Gauss hypergeometric function $z \mapsto {}_2F_1(a, b; c; z)$ depends on the three parameters $a, b, c \in \mathbb{C}$, $c \neq \{0, -1, -2, \ldots\}$, and is defined for $z \in E$ by the series expansion

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(\sigma)_k$ is the Pochhammer symbol defined, in terms of Gamma function, by

$$(\sigma)_k = \frac{\Gamma(\sigma + k)}{\Gamma(\sigma)} = \begin{cases} 1, & k = 0, \\ \sigma(\sigma + 1) \cdots (\sigma + k - 1), & k \in \mathbb{N} = \{1, 2, \ldots\}. \end{cases}$$

Define the incomplete beta function $\varphi(a, b)$, for $a, b \in \mathbb{R}$, by

$$\varphi(a, c; z) = z\,{}_2F_1(1, a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1}, \quad c \neq 0, -1, -2, \ldots; \ z \in E.$$

The fractional derivative of order α is defined [13] for a function f by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z - t)^{\alpha}}\, dt, \quad 0 \leqslant \alpha < 1,$$

where f is an analytic function in a simply connected domain of the z-plane containing the origin and the multiplicity of $(z - t)^{-\alpha}$ is removed by requiring $\log(z - t)$ to be real when $z - t > 0$.

Using D_z^{α}, Owa and Srivastava [14] introduced the operator $\Omega^{\alpha} : A \to A$, which is known as an extension of fractional derivative and fractional integral, as follows:

$$\begin{aligned} \Omega^{\alpha} f(z) &= \Gamma(2 - \alpha) z^{\alpha} D_z^{\alpha} f(z), \quad \alpha \neq 2, 3, 4, \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k + 1)\Gamma(2 - \alpha)}{\Gamma(k + 1 - \alpha)} \alpha_k z^k \qquad (1.2) \\ &= \varphi(2, 2 - \alpha; z) * f(z). \qquad (1.3) \end{aligned}$$

Note that $\Omega^0 f(z) = f(z)$.

The linear fractional differential operator $D_{\lambda}^{n,\alpha} : A \to A$ was defined by the authors in [3] as follows:

$$\begin{aligned} &D_{\lambda}^{0,0} f(z) = f(z), \\ &D_{\lambda}^{1,\alpha} f(z) = (1 - \lambda)\Omega^{\alpha} f(z) + \lambda z\big(\Omega^{\alpha} f(z)\big)' = D_{\lambda}^{\alpha}\big(f(z)\big), \quad 0 \leqslant \alpha < 1, \ \lambda \geqslant 0, \qquad (1.4) \\ &D_{\lambda}^{2,\alpha} f(z) = D_{\lambda}^{\alpha}\big(D_{\lambda}^{1,\alpha} f(z)\big), \\ &\quad \vdots \\ &D_{\lambda}^{n,\alpha} f(z) = D_{\lambda}^{\alpha}\big(D_{\lambda}^{n-1,\alpha} f(z)\big), \quad n \in \mathbb{N}. \qquad (1.5) \end{aligned}$$

If f is given by (1.1), then by (1.2), (1.4) and (1.5), we see that

$$D_\lambda^{n,\alpha} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} \big(1+\lambda(k-1)\big) \right)^n a_k z^k, \quad n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}. \tag{1.6}$$

From (1.3) and (1.6), $D_\lambda^{n,\alpha} f(z)$ can be written, in terms of convolution as

$$D_\lambda^{n,\alpha} f(z) = \underbrace{\big[\big(\varphi(2, 2-\alpha; z) * g_\lambda(z)\big) * \cdots * \big(\varphi(2, 2-\alpha; z) * g_\lambda(z)\big)\big]}_{n\text{-times}} * f(z), \tag{1.7}$$

where

$$g_\lambda(z) = \frac{z-(1-\lambda)z^2}{(1-z)^2} = z + \sum_{k=2}^{\infty} \big[1+\lambda(k-1)\big]^n z^k.$$

When $\alpha = 0$, we get Al-Oboudi differential operator [2], when $\alpha = 0$ and $\lambda = 1$, we get Sălăgean differential operator [20] and when $n = 1$ and $\lambda = 0$, we get Owa–Srivastava fractional differential operator [14].

Using the operator $D_\lambda^{n,\alpha}$, authors define in [3] the classes $UCV_{\alpha,\lambda}^n(\beta,\gamma)$ and $SP_{\alpha,\lambda}^n(\beta,\gamma)$ as follows:

$$\mathrm{Re}\left\{1 + \frac{z(D_\lambda^{n,\alpha} f(z))''}{(D_\lambda^{n,\alpha} f(z))'}\right\} > \beta \left| \frac{z(D_\lambda^{n,\alpha} f(z))''}{(D_\lambda^{n,\alpha} f(z))'} \right| + \gamma, \tag{1.8}$$

where $\beta \geqslant 0$, $-1 \leqslant \gamma < 1$, $\beta + \gamma \geqslant 0$, $n \in \mathbb{N}_0$, $0 \leqslant \alpha < 1$, $\lambda \geqslant 0$.

Let $SP_{\alpha,\lambda}^n(\beta,\gamma)$ be the corresponding class consisting of $f \in A$ satisfying

$$\mathrm{Re}\left\{\frac{z(D_\lambda^{n,\alpha} f(z))'}{D_\lambda^{n,\alpha} f(z)}\right\} > \beta \left| \frac{z(D_\lambda^{n,\alpha} f(z))'}{D_\lambda^{n,\alpha} f(z)} - 1 \right| + \gamma, \tag{1.9}$$

where $\beta \geqslant 0$, $-1 \leqslant \gamma < 1$, $\beta + \gamma \geqslant 0$, $n \in \mathbb{N}_0$, $0 \leqslant \alpha < 1$, $\lambda \geqslant 0$.

It is clear that

$$f \in UCV_{\alpha,\lambda}^n(\beta,\gamma) \quad \Leftrightarrow \quad zf' \in SP_{\alpha,\lambda}^n(\beta,\gamma). \tag{1.10}$$

These classes generalize various other classes investigated earlier by (for example) Goodman [7], Rønning [16,17], Kanas and Wisniowska [8,9], and Srivastava and Mishra [22].

For special values of the parameters α and β, we refer to the following classes in [3]

$$UCV_{0,\lambda}^n(\beta,\gamma) \equiv UCV_\lambda^n(\beta,\gamma);$$
$$SP_{0,\lambda}^n(\beta,\gamma) \equiv SP_\lambda^n(\beta,\gamma);$$
$$UCV_{\alpha,\lambda}^n(0,\gamma) \equiv CV_{\alpha,\lambda}^n(\gamma);$$
$$SP_{\alpha,\lambda}^n(0,\gamma) \equiv ST_{\alpha,\lambda}^n(\gamma),$$

which have not been studied.

In [3], we proved that for $0 \leqslant \lambda \leqslant \frac{1+\beta}{1-\gamma}$,

$$SP_{\alpha,\lambda}^{n+1}(\beta,\gamma) \subseteq SP_{\alpha,\lambda}^n(\beta,\gamma) \subseteq SP(\beta,\gamma),$$
$$UCV_{\alpha,\lambda}^{n+1}(\beta,\gamma) \subseteq UCV_{\alpha,\lambda}^n(\beta,\gamma) \subseteq UCV(\beta,\gamma).$$

Denote by $\mathcal{P}(P_{\beta,\gamma})$ ($\beta \geqslant 0$, $-1 \leqslant \gamma < 1$, $\beta+\gamma \geqslant 0$), the family of functions p, such that $p \in \mathcal{P}$ and $p \prec P_{\beta,\gamma}$ in E, where $\mathcal{P}$ denotes the well-known class of Caratheodory functions and the function $P_{\beta,\gamma}$ maps the unit disk conformally onto the conic domain $R_{\beta,\gamma}$ such that $1 \in R_{\beta,\gamma}$ and $\partial R_{\beta,\gamma}$ is a curve defined by the equality

$$\partial R_{\beta,\gamma} = \left\{u + iv;\ u^2 = \left(\beta\sqrt{(u-1)^2+v^2} + \gamma\right)^2\right\}.$$

From elementary computations, we see that $\partial R_{\beta,\gamma}$ represents the conic sections symmetric about the real axis. Thus $R_{\beta,\gamma}$ is an elliptic domain for $\beta > 1$, a parabolic domain for $\beta = 1$, a hyperbolic domain for $0 < \beta < 1$ and a right half-plane $u > \gamma$ for $\beta = 0$.

The functions, which play the role of extremal functions of the class $\mathcal{P}(P_{\beta,\gamma})$, were obtained in [1] as follows:

$$P_{\beta,\gamma}(z) = \begin{cases} \frac{1+(1-2\gamma)z}{1-z}, & \beta = 0, \\ \frac{1-\gamma}{1-\beta^2}\cos\{\frac{2}{\pi}(\cos^{-1}\beta) i \operatorname{Log}\frac{1+\sqrt{z}}{1-\sqrt{z}}\} - \frac{\beta^2-\gamma}{1-\beta^2}, & 0 < \beta < 1, \\ 1 + \frac{2(1-\gamma)}{\pi^2}(\operatorname{Log}\frac{1+\sqrt{z}}{1-\sqrt{z}})^2, & \beta = 1, \\ \frac{1-\gamma}{\beta^2-1}\sin(\frac{\pi}{2V(t)}\int_0^{\ell(z)/\sqrt{t}} \frac{1}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\,dx) + \frac{\beta^2-\gamma}{\beta^2-1}, & \beta > 1, \end{cases} \tag{1.11}$$

where $\ell(z)=\frac{z-\sqrt{t}}{1-\sqrt{t}z}$, $t\in(0,1)$, $z\in E$, and t is chosen such that $\beta=\cosh\frac{\pi V'(t)}{4V(t)}$, V is Legendre's complete elliptic integral of the first kind and V' is complementary integral of V.

Geometric interpretation: From (1.8) and (1.9), $f\in UCV^n_{\alpha,\lambda}(\beta,\gamma)$ and $f\in SP^n_{\alpha,\lambda}(\beta,\gamma)$ if and only if $p(z)=1+\frac{z(D^{n,\alpha}_\lambda f(z))''}{(D^{n,\alpha}_\lambda f(z))'}$ and $p(z)=\frac{z(D^{n,\alpha}_\lambda f(z))'}{D^{n,\alpha}_\lambda f(z)}$, respectively, take all values in conic domain $R_{\beta,\gamma}$ which is included in the right half-plane. We may rewrite the conditions (1.8) and (1.9) in the form

$$p(z)\prec P_{\beta,\gamma}(z), \tag{1.12}$$

where the functions $P_{\beta,\gamma}$ are given in (1.11).

In [3], basic properties of the classes $UCV^n_{\alpha,\lambda}(\beta,\gamma)$ and $SP^n_{\alpha,\lambda}(\beta,\gamma)$ are studied, such as inclusion relations and coefficient bounds. In this paper, several interesting subordination results are derived for these classes, which yield sharp distortion, rotation theorems and Koebe domain.

2. Subordination theorems and consequence

For analytic function $f(0)=0$, and $f'(0)\neq 0$ its order of convexity with respect to zero is defined by

$$\kappa(f)=1+\inf_{z\in E}\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}\in[-\infty,1].$$

The function f is convex if $\kappa f()\geqslant 0$.

We need the following lemmas.

Lemma 2.1. *(See [18].) Let f and g be convex univalent functions in E. Then so is $f*g$.*

Lemma 2.2. *(See [19].) Let F and G be convex univalent functions in E. Also, let $f\prec F$ and $g\prec G$. Then $f*g\prec F*G$.*

Lemma 2.3. *(See [10].) If $-1<\alpha\leqslant b\leqslant c$ and $abc\neq 0$, then*

$$1-\frac{(a+1)(b+1)}{c+b+2}\leqslant\kappa\big({}_2F_1(a,b;c,z)\big)=1-\frac{{}_2F_1''(a,b;c;-1)}{{}_2F_1'(a,b;c;-1)}\leqslant 1-\frac{(a+1)(b+1)}{2(c+1)}.$$

Lemma 2.4. *The function $\frac{h_\lambda}{z}$ is convex univalent, where*

$$h_\lambda(z)=z+\sum_{k=2}^{\infty}\frac{1}{1+\lambda(k-1)}z^k. \tag{2.1}$$

Proof. For $\lambda=0$, $\frac{h_0(z)}{z}=\frac{1}{(1-z)}$, which is convex univalent.

For $\lambda>0$, we can rewrite $\frac{h_\lambda(z)}{z}$ as follows:

$$\frac{h_\lambda(z)}{z}={}_2F_1\left(1,\frac{1}{\lambda};1+\frac{1}{\lambda};z\right)={}_2F_1\left(\frac{1}{\lambda},1;1+\frac{1}{\lambda};z\right).$$

We have two cases:

Case 1. For $\lambda\leqslant 1$, and $a=1$, $b=\frac{1}{\lambda}$, $c=1+\frac{1}{\lambda}$, we have $-1<a\leqslant b\leqslant c$. Applying Lemma 2.1, we have

$$1-\frac{2(1+1/\lambda)}{3+2/\lambda}\leqslant\kappa\left({}_2F_1\left(1,\frac{1}{\lambda};1+\frac{1}{\lambda};z\right)\right)\leqslant 1-\frac{(1+1/\lambda)}{(2+1/\lambda)}.$$

The left-hand side inequality is greater than or equal to zero, thus $\kappa({}_2F_1(1,\frac{1}{\lambda};1+\frac{1}{\lambda};z))\geqslant 0$.

Case 2. For $\lambda\geqslant 1$, and $a=\frac{1}{\lambda}$, $b=1$, $c=1+\frac{1}{\lambda}$, we have $-1<a\leqslant b\leqslant c$. Applying Lemma 2.1, we get

$$1-\frac{2(1+1/\lambda)}{4+1/\lambda}\leqslant\kappa\left({}_2F_2\left(1,\frac{1}{\lambda};\frac{1}{\lambda}+1;z\right)\right)\leqslant 1-\frac{(1+1/\lambda)}{(2+1/\lambda)}.$$

The left-hand side inequality is greater than or equal to zero for $\lambda\geqslant 1$, thus $\kappa({}_2F_1(1,\frac{1}{\lambda};1+\frac{1}{\lambda};z))\geqslant 0$.

In both cases, we get that $\frac{h_\lambda(z)}{z}$ is convex univalent. $\square$

Theorem 2.1. *Let $M(z)=1+m_1z+m_2z^2+\cdots$ be a convex and univalent function in E and let $f\in A$. If*

$$\frac{D_\lambda^{n,\alpha}f(z)}{z}\prec M(z), \tag{2.2}$$

then

$$\frac{f(z)}{z}\prec\left(\frac{1}{z}\right)\left\{\Phi_{\alpha,\lambda}(z)*zM(z)\right\},$$

where

$$\Phi_{\alpha,\lambda}(z)=\big(\varphi(2-\alpha,2;z)*h_\lambda(z)\big)*\cdots*\big(\varphi(2-\alpha,2;z)*h_\lambda(z)\big). \tag{2.3}$$

Proof. Let h_λ be defined by (2.1). By using (1.7) and (2.3), we can see that

$$\frac{f(z)}{z}=\underbrace{\left[\frac{\varphi(2-\alpha,2;z)*h_\lambda(z)}{z}*\cdots*\frac{\varphi(2-\alpha,2;z)*h_\lambda(z)}{z}\right]}_{n\text{-times}}*\frac{D_\lambda^{n,\alpha}f(z)}{z}=\frac{\Phi_{\alpha,\lambda}(z)}{z}*\frac{D_\lambda^{n,\alpha}f(z)}{z}. \tag{2.4}$$

In [5], it is shown that the function $\frac{\varphi(2-\alpha,2;z)}{z}$ is convex univalent in E.

Also, by using Lemma 2.4, $\frac{h_\lambda(z)}{z}$ is convex univalent in E, and applying Lemma 2.1 n-times, we get that $\frac{\Phi_{\alpha,\lambda}(z)}{z}$ is convex univalent.

From (2.2), (2.4) and using Lemma 2.2 with $g(z)=G(z)=\frac{\Phi_{\alpha,\lambda}(z)}{z}$, we get

$$\frac{f(z)}{z}\prec\frac{\Phi_{\alpha,\lambda}(z)}{z}*M(z)=\frac{1}{z}\left\{\Phi_{\alpha,\lambda}(z)*zM(z)\right\}.$$

By considering the function $f(z)=\Phi_{\alpha,\lambda}(z)*zM(z)$, we can show that the result is best possible. □

Remark 2.1. For $n=1$, and $\lambda=0$, Theorem 2.1 reduces to a result obtained in [11].

Remark 2.2. For special parameters α and λ, we get new subordination results for Sălăgean differential operator ($\alpha=0$, $\lambda=1$), and Al-Oboudi differential operator ($\alpha=0$).

Let

$$F'_{\beta,\gamma}(z)=\left(\frac{1}{z}\right)\left\{\Phi_{\alpha,\lambda}(z)*zQ'_{\beta,\gamma}(z)\right\}, \tag{2.5}$$

where

$$Q'_{\beta,\gamma}(z)=\exp\int_0^z\frac{P_{\beta,\gamma}(\xi)-1}{\xi}\,d\xi, \tag{2.6}$$

and $\Phi_{\alpha,\lambda}$, $P_{\beta,\gamma}$ are defined by (2.3) and (1.11), respectively.

Using (2.5) and (2.6), we prove the next results.

Theorem 2.2. *Let $\{(0\leqslant\beta<1$ and $\frac{1}{2}\leqslant\gamma<1)$ or $(\beta\geqslant1, 0\leqslant\gamma<1)\}$. Then the functions $Q'_{\beta,\gamma}$ and $F'_{\beta,\gamma}$ defined, respectively, by* (2.6) *and* (2.5) *are convex univalent in E.*

Proof. By direct calculation, we can show that

$$\operatorname{Re}\left\{1+\frac{zQ'''_{\beta,\gamma}(z)}{Q''_{\beta,\gamma}(z)}\right\}=\operatorname{Re}\left\{P_{\beta,\gamma}(z)-1\right\}+\operatorname{Re}\left\{\frac{zP'_{\beta,\gamma}(z)}{P_{\beta,\gamma}(z)-1}\right\}.$$

Note that $\frac{1}{P_1}(P_{\beta,\gamma}(z)-1)$ is in CV, P_1 is the coefficient of the function $P_{\beta,\gamma}$, for $k=1$ in the series representation, which implies (see [15, p. 45])

$$\operatorname{Re}\left\{\frac{zP'_{\beta,\gamma}(z)}{P_{\beta,\gamma}(z)-1}\right\}>\frac{1}{2}.$$

By virtue of the properties of the domain $R_{\beta,\gamma}$, we have

$$\operatorname{Re}\left\{P_{\beta,\gamma}(z)\right\}>\frac{\beta+\gamma}{1+\beta}.$$

Also, for $\{(0 \leqslant \beta < 1 \text{ and } \frac{1}{2} \leqslant \gamma < 1) \text{ or } (\beta \geqslant 1,\ 0 \leqslant \gamma < 1)\}$, we have

$$\operatorname{Re}\{P_{\beta,\gamma}(z) - 1\} > -\frac{1}{2}.$$

Thus, we have

$$\operatorname{Re}\left\{1 + \frac{zQ'''_{\beta,\gamma}(z)}{Q''_{\beta,\gamma}(z)}\right\} > 0.$$

Therefore $Q'_{\beta,\gamma}$ is convex univalent.

Next, we observe that

$$F'_{\beta,\gamma}(z) = \frac{\Phi_{\alpha,\lambda}(z)}{z} * Q'_{\beta,\gamma}(z).$$

From the proof of Theorem 2.1, we get that $\frac{\Phi_{\alpha,\lambda}(z)}{z}$ is convex univalent. Therefore, by Lemma 2.1, $F'_{\beta,\gamma}$ is convex univalent. □

Theorem 2.3. *Let $\{(0 \leqslant \beta < 1$ and $\frac{1}{2} \leqslant \gamma < 1)$ or $(\beta \geqslant 1,\ 0 \leqslant \gamma < 1)\}$ and let f be in the class $SP^n_{\alpha,\lambda}(\beta,\gamma)$. Then*

$$\frac{f(z)}{z} \prec F'_{\beta,\gamma}(z),$$

where $F'_{\beta,\gamma}$ is defined by (2.5)*. The result is best possible.*

Proof. Let $f \in SP^n_{\alpha,\lambda}(\beta,\gamma)$. Then by (1.12),

$$\frac{z(D^{n,\alpha}_\lambda f(z))'}{D^{n,\alpha}_\lambda f(z)} \prec P_{\beta,\gamma}(z),$$

which implies

$$\frac{z(D^{n,\alpha}_\lambda f(z))'}{D^{n,\alpha}_\lambda f(z)} - 1 \prec P_{\beta,\gamma}(z) - 1.$$

Note that $P_{\beta,\gamma}(z) - 1$ is a univalent convex function in E. Using a result of Goluzin [6] (see also [15, p. 50]), we have

$$\operatorname{Log} \frac{D^{n,\alpha}_\lambda f(z)}{z} \prec \int_0^z \frac{P_{\beta,\gamma}(\xi) - 1}{\xi}\, d\xi.$$

Thus there exists a function $w \in A$ satisfying $w(0) = 0$ and $|w(z)| < 1$, $z \in E$ such that

$$\operatorname{Log} \frac{D^{n,\alpha}_\lambda f(z)}{z} = \int_0^z \frac{P_{\beta,\gamma}(w(\xi)) - 1}{w(\xi)}\, dw(\xi),$$

which is equivalent to

$$\frac{D^{n,\alpha}_\lambda f(z)}{z} \prec \exp\left(\int_0^z \frac{P_{\beta,\gamma}(\xi) - 1}{\xi}\, d\xi\right) := Q'_{\beta,\gamma}(z). \qquad \square$$

Remark 2.3. 1. For $n = 0, \alpha = 0, \beta = 1$, and $\gamma = 0$, Theorem 2.2 reduces to a result obtained in [12].

2. For $n = 1, \lambda = 0$, and $\beta = 0$, Theorem 2.3 reduces to a result obtained in [21].

3. For $n = 1, \lambda = 0, \beta = 1$ and $\gamma = 0$, Theorem 2.3 reduces to a result obtained in [22].

Theorem 2.4. *Let $\{(0 \leqslant \beta < 1$ and $\frac{1}{2} \leqslant \gamma < 1)$ or $(\beta \geqslant 1,\ 0 \leqslant \gamma < 1)\}$ and let f be in the class $SP^n_{\alpha,\gamma}(\beta,\gamma)$. Then*

$$F'_{\beta,\gamma}(-r) \leqslant \left|\frac{f(z)}{z}\right| \leqslant F'_{\beta,\gamma}(r), \quad |z| = r < 1 \tag{2.7}$$

and

$$\left|\operatorname{Arg} \frac{f(z_0)}{z_0}\right| \leqslant \max_{|z|=r}\{\operatorname{Arg} F'_{\beta,\gamma}(z)\}, \quad |z_0| < 1, \tag{2.8}$$

where $F'_{\beta,\gamma}$ is defined by (2.5)*. These inequalities are sharp.*

Proof. Let $f \in SP^n_{\alpha,\lambda}(\beta,\gamma)$. Then by Theorem 2.3 and Lindelof's principle of subordination, we get

$$\inf_{|z|\leqslant r} \operatorname{Re}\{F'_{\beta,\gamma}(z)\} \leqslant \inf_{|z|\leqslant r} \operatorname{Re}\left\{\frac{f(z)}{z}\right\} \leqslant \sup_{|z|\leqslant r} \operatorname{Re}\left\{\frac{f(z)}{z}\right\} \leqslant \left|\frac{f(z)}{z}\right| \leqslant \sup\left|\frac{f(z)}{z}\right| \leqslant \sup_{|z|\leqslant r} \operatorname{Re}\{F'_{\beta,\gamma}(z)\}. \tag{2.9}$$

Since $F'_{\beta,\gamma}$ is convex univalent and has real coefficient, $F'_{\beta,\gamma}(E)$ is a convex domain symmetric with respect to real axis. Hence,

$$\inf_{|z|\leqslant r} \operatorname{Re}\{F'_{\beta,\gamma}(z)\} = \inf_{-r\leqslant x\leqslant r} \{F'_{\beta,\gamma}(x)\} = F'_{\beta,\gamma}(-r),$$
$$\sup_{|z|\leqslant r} \operatorname{Re}\{F'_{\beta,\gamma}(z)\} = \sup_{-r\leqslant x\leqslant r} \{F'_{\beta,\gamma}(x)\} = F'_{\beta,\gamma}(r).$$

Thus (2.9) gives the assertion (2.7) of Theorem 2.4.

Similarly, from Theorem 2.3, we get the rotation assertion (2.8). Equality holds true in (2.7) and (2.8) for some $z \neq 0$, $z_0 \neq 0$, respectively, if and only if f is a rotation of $f_{\beta,\gamma}(z) = zF'_{\beta,\gamma}(z)$. □

Since coefficient bounds are sharp for $\beta = 0$, we get a better result for the function $f \in ST^n_{\alpha,\lambda}(\gamma)$ as follows:

Theorem 2.5. *Let f be in the class $ST^n_{\alpha,\lambda}(\gamma)$. Then*

$$|f(z)| \leqslant rF'_{0,\gamma}(r), \quad |z| = r < 1. \tag{2.10}$$

Furthermore, if $\frac{1}{2} \leqslant \gamma < 1$, then

$$\left|\frac{f(z)}{z}\right| \geqslant \operatorname{Re}\left\{\frac{f(z)}{z}\right\} \geqslant F'_{0,\gamma}(-r). \tag{2.11}$$

Both estimates (2.10) *and* (2.11) *are sharp if f is a rotation of $f_{0,\gamma}(z) = zF'_{0,\gamma}(z)$, where*

$$F'_{0,\gamma}(z) = \frac{1}{z}\left(\Phi_{\alpha,\lambda}(z) * \frac{z}{(1-z)^{2(1-\gamma)}}\right), \tag{2.12}$$

and $\Phi_{\alpha,\lambda}$ is defined by (2.3).

Proof. By direct calculations, we find

$$zQ'_{0,\gamma}(z) = \frac{z}{(1-z)^{2(1-\gamma)}} = z + \sum_{k=2}^{\infty} \prod_{j=2}^{k} (j-2\gamma)\frac{z^k}{(k-1)!}.$$

Now let $f \in ST^n_{\alpha,\lambda}(\gamma)$. In [3], we proved that

$$|a_k| \leqslant \left(\frac{\Gamma(k+1-\alpha)}{\Gamma(k+1)\Gamma(2-\alpha)(1+\lambda(k-1))}\right)^n \prod_{j=2}^{k} \frac{(j-2\gamma)}{(k-1)!}$$

and we mentioned that the result is sharp. Then we have

$$\begin{aligned}
|f(z)| &\leqslant |z| + \sum_{k=2}^{\infty} |a_k||z^k| \leqslant |z| + \sum_{k=2}^{\infty}\left(\frac{\Gamma(k+1-\alpha)}{\Gamma(k+1)\Gamma(2-\alpha)(1+\lambda(k-1))}\right)^n \prod_{j=2}^{k}(j-2\gamma)\frac{|z^k|}{(k-1)!} \\
&= \left(|z| + \sum_{k=2}^{\infty}\left(\frac{\Gamma(k+1-\alpha)}{\Gamma(k+1)\Gamma(2-\alpha)(1+\lambda(k-1))}\right)^n |z^k|\right) * \left(|z| + \sum_{k=2}^{\infty}\prod_{j=2}^{k}(j-2\gamma)\frac{|z^k|}{k!}\right) \\
&= rF'_{0,\gamma}(r),
\end{aligned}$$

which yields (2.10).

Next, suppose that $\frac{1}{2} \leqslant \gamma < 1$. Then, by Theorem 2.5 and inequality (2.9), we get (2.11). □

Corollary 2.1 *(Koebe domain). Let $\{(0 \leqslant \beta < 1$ and $\frac{1}{2} \leqslant \gamma < 1)$ or $(\beta \geqslant 1,\ 0 \leqslant \gamma < 1)\}$ and let f be in the class $SP^n_{\alpha,\gamma}(\beta,\gamma)$. Then for $0 \leqslant \lambda \leqslant \frac{1+\beta}{1-\gamma}$,*

$$K\big(SP^n_{\alpha,\lambda}(\beta,\gamma)\big) = \{w\colon |w| \leqslant F'_{\beta,\gamma}(-1) = -f_{\beta,\gamma}(-1)\} \subseteq f(E).$$

The result is sharp for a rotation of $f_{\beta,\gamma}(z) = zF'_{\beta,\gamma}(z)$.

By virtue of (1.10) and Theorems 2.3, 2.4, we get the following results.

Corollary 2.2. *Let $\{(0 \leqslant \beta < 1$ and $\frac{1}{2} \leqslant \gamma < 1)$ or $(\beta \geqslant 1, 0 \leqslant \gamma < 1)\}$ and let f be in the class $UCV^n_{\alpha,\lambda}(\beta,\gamma)$. Then*

$$f'(z) \prec F'_{\beta,\gamma}(z), \tag{2.13}$$

$$F'_{\beta,\gamma}(-r) \leqslant \left|f'(z)\right| \leqslant F'_{\beta,\gamma}(r), \tag{2.14}$$

$$\left|\operatorname{Arg} f'(z_0)\right| \leqslant \max_{|z|=r}\left\{\operatorname{Arg} F'_{\beta,\gamma}(z)\right\}, \quad |z_0| < 1, \tag{2.15}$$

where $F'_{\beta,\gamma}$ is defined by (2.5). The result (2.13) is best possible, and equality holds in (2.14), and (2.15) when $z \neq 0$, $z_0 \neq 0$, respectively, if and only if f is a rotation of the function $F_{\beta,\gamma}$.

Corollary 2.3. *Let f be in the class $CV^n_{\alpha,\lambda}(\gamma)$. Then*

$$\left|f'(z)\right| \leqslant F'_{0,\gamma}(r), \quad |z| = r. \tag{2.16}$$

Furthermore, if $\frac{1}{2} \leqslant \gamma < 1$, then

$$\left|f'(z)\right| \geqslant \operatorname{Re}\left\{f'(z)\right\} \geqslant F'_{0,\gamma}(-r). \tag{2.17}$$

Both estimates (2.16) and (2.17) are sharp if f is a rotation of $F_{0,\gamma}$, where $F'_{0,\gamma}$ is defined by (2.12).

By using standard techniques and Corollary 2.2, we obtain the distortion theorem and Koebe domain of the class $UCV^n_{\alpha,\lambda}(\beta,\gamma)$.

Corollary 2.4. *Let $\{(0 \leqslant \beta < 1$ and $\frac{1}{2} \leqslant \gamma < 1)$ or $(\beta \geqslant 1, 0 \leqslant \gamma < 1)\}$ and f be in the class $UCV^n_{\alpha,\lambda}(\beta,\gamma)$. Then*

$$-F_{\beta,\gamma}(-r) \leqslant \left|f(z)\right| \leqslant F_{\beta,\gamma}(r),$$

and for $0 \leqslant \lambda \leqslant \frac{1+\beta}{1-\gamma}$,

$$K\left(UCV^n_{\alpha,\lambda}(\beta,\gamma)\right) = \left\{w\colon |w| \leqslant -F_{\beta,\gamma}(-1)\right\} \subseteq f(E).$$

The result is sharp for a rotation of $F_{\beta,\gamma}$.

Corollary 2.5. *Let f be in the class $CV^n_{\alpha,\lambda}(\gamma)$. Then*

$$\left|f(z)\right| \leqslant F_{0,\gamma}(r), \quad |z| = r. \tag{2.18}$$

Furthermore, if $\frac{1}{2} \leqslant \gamma < 1$, then

$$\left|f(z)\right| \geqslant -F_{0,\gamma}(-r). \tag{2.19}$$

Both estimates (2.18) and (2.19) are sharp if f is a rotation of $F_{0,\gamma}$.

Remark 2.4. 1. For $n = 0$, $\alpha = 0$, $\beta = 1$, and $\gamma = 0$, Corollaries 2.2 and 2.4 reduce to results obtained in [12].
2. For $n = 0$, $\alpha = 0$, and $\gamma = 0$, Corollaries 2.2 and 2.4 reduce to results obtained in [8].
3. For $n = 0$, $\alpha = 0$, $\beta = 1$, and $\gamma = 0$, Theorem 2.4 (inequality (2.7)) and Corollary 2.1 reduce to results obtained in [16].
4. For $n = 1$, $\lambda = 0$, $\beta = 1$, and $\gamma = 0$, Theorem 2.4 and Corollary 2.2 reduce to results obtained in [22].
5. For $n = 1$, $\lambda = 0$, and $\beta = 0$, Theorem 2.5 reduces to a result obtained in [21].
6. For $n = 0$, $\alpha = 0$, and $\gamma = 0$, Theorem 2.4 (inequality (2.7)) reduces to a result obtained in [9].

Remark 2.5. For special values of the parameters n, α, λ and β in Theorems 2.3–2.5 and Corollaries 2.1–2.5, we obtain subordination results, distortion theorems, rotation theorems and Koebe domains for the classes $SP(\beta,\gamma)$, $UCV(\beta,\gamma)$, $SP^1_{\alpha,0}(\beta,\gamma)$, $UCV^1_{\alpha,0}(\beta,\gamma)$, $SP^n_{\lambda},(\beta,\gamma)$, $UCV^n_{\lambda}(\beta,\gamma)$, $ST^n_{\alpha,\lambda}(\gamma)$, and $CV^n_{\alpha,\lambda}(\gamma)$ which are new results.

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